# Dialogue Games for Classical Logic

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**Abstract.** We define a class of dialogue games and prove that existence of winning strategies for the Proponent in this class of games corresponds to validity in classical propositional logic. Many authors have stated similar results without actually proving the correspondence. We modify the games used for intuitionistic logic given by Fermüller [3]. We employ standard dialogue games and a standard sequent calculus for classical logic. The result is a simple correspondence between dialogue games and classical logic.

## 1 Introduction

Dialogue games as a semantics for intuitionistic logic (IL) were developed by Lorenzen in the 1950s as an alternative to the operative approach to logic [8, 9]. A *dialogue game* is a finitary open two-person zero-sum game between the Proponent **P** and the Opponent **O**. Lorenzen's goal was to isolate a certain class of dialogue games such that **P** has a winning strategy for the dialogue game beginning with $\varphi$ iff $\varphi$ is a theorem of IL.

The first recognized successful proof for IL was given by Felscher [2]. Felscher's proof, though correct, is both complicated, with its introduction of the notion of protableaux, and difficult to understand. Fermüller [3] provides a variant of the E rules which allows him to give a simpler proof of the correspondence, without intermediate recourse to protableaux (the proof also appears in [4]). Our goal in this paper is to modify Fermüller's games and prove that these modified games characterize classical validity; we do this in §3.

## 2 Previous work

Numerous classes of dialogue games for classical logic have been defined in the literature, but with few exceptions, the correspondence between the class of games

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and classical validity is merely asserted, and not proved [1, §2], [5, pp. 352–53], [6, pp. 217–18], [7, p. 305], [9, p. 194], [10, §1.5], [11, §2], [12, passim], [14, p. 152]. The three exceptions are due to Fermüller and Sørensen & Urzyczyn. In [3], Fermüller defines classes of "parallel" dialogue games which capture various intermediate logics, including classical logic, characterized by various hypersequent calculi. In [13], Sørensen and Urzyczyn offer an elegant and compact correspondence between Felscher's dialogues for classical logic and a "dialogue-inspired" variant of the sequent calculus **LK** for classical logic that they call **LKD**. The calculus **LKD** has only three rules: a left-rule, a right-rule, and one structural rule, cut, which can be eliminated [13, Cor. 3.4]. The left and right rules have side conditions that are built directly from the dialogue rules for attacks and defenses of formulas; this justifies the presence of only two logical rules in the calculus, rather than the customary array of left and right rules for each of the connectives.

Our result in this paper improves on both Fermüller's and Sørensen and Urzyczyn's proofs. We show that, in the classical case, Fermüller's use of parallel dialogue games and hypersequents can be avoided, and that a correspondence with a standard sequent calculus, rather than Sørensen and Urzyczyn's "dialogue-inspired" version can be established.

## 3 Classical dialogical logic

Our language is a basic propositional language with a designated atom $\bot$ (*falsum*). We define $\neg\varphi$ as $\varphi \to \bot$. We will also make use of so-called *symbolic attacks*, $\wedge_L$, $\wedge_R$, and ?.

Dialogue games are specified by two types of rules, *particle* (local) rules and *structural* (global) rules. Particle rules (see Table 1) give the attack and defense conditions for each type of formula; note that we allow attacks on atoms, but these attacks cannot be defended. Structural rules define which sequences of dialogical moves will count as legal dialogues. A *dialogue* is a sequence of attacks and defenses that begins with a finite (possibly empty) multiset $\Pi$ of formulas that are *initially granted* by **O** and a finite (nonempty) multiset $\Delta$ of formulas that are *initially disputed* by **O**. Formulas that have been initially granted by **O** can be attacked by **P** at any time, and formulas that are initially disputed by **O** can be asserted as a defense by **P** at any time. In the case where $\Delta$ is a singleton, we can understand the game as beginning with an assertion of $\Delta$ by **P**, with the first move then being an attack on $\Delta$ by **O**.

**Definition 1 (CL structural rules).**

**Start** *The first move of the dialogue is carried out by* **O** *and consists in an attack on (the unique) initially disputed formula* $\varphi$.

**Alternation** *Moves strictly alternate between players* **O** *and* **P**.

**Atom** *Atomic formulas, including* $\bot$*, may be stated by both players, but only* **O** *can attack them.*

**E** *Each move of* **O** *reacts directly to the immediately preceding move by* **P**.

| **Assertion** | **Attack** | **Response** |
|---|---|---|
| $p$ (atomic) | ? | — |
| $\varphi \wedge \psi$ | $\wedge_L$ | $\varphi$ |
| | $\wedge_R$ | $\psi$ |
| $\varphi \vee \psi$ | ? | $\varphi$ or $\psi$ |
| $\varphi \to \psi$ | $\varphi$ | $\psi$ |

**Table 1.** Particle rules for dialogue games

**Definition 2 (Active formula).** *The most recent formula which* **P** *has asserted that* **O** *must attack in the next round is the* active formula, *if it exists.*

**Definition 3 (Winning conditions (for P)).**

$W^{\mathsf{CL}}$ *The game ends with* **P** *winning if* $\Pi \cap \Delta \neq \emptyset$.
$W_{\bot}$ *The game ends when* $\bot$ *is granted.*

To establish the correspondence between classical validity and (existence of) winning strategies, we use a variant of the sequent calculus system **GKcp** [15] for classical propositional logic, which is itself a variant of the standard contraction- and weakening-friendly formulation of **LK** that copies the principal formula into the premise (or premises).

**Definition 4 (The system GKcp′).** *Derivable objects are sequents* $\Pi \Rightarrow \Delta$, *where* $\Pi$ *and* $\Delta$ *are multisets of formulas. The sequent system* **GKcp′** *is specified by the axioms and rules in Figure 1, together with the usual weakening and contraction rules on both the left and the right, as well as cut.*

Axioms: $\varphi, \Pi \Rightarrow \Delta, \varphi$ and $\bot, \Pi \Rightarrow \Delta$

$$\vee\text{L}\ \frac{A \vee B, A, \Pi \Rightarrow \Delta \qquad A \vee B, B, \Pi \Rightarrow \Delta}{A \vee B, \Pi \Rightarrow \Delta} \qquad \frac{\Pi \Rightarrow \Delta, A \vee B, A, B}{\Pi \Rightarrow \Delta, A \vee B}\ \vee\text{R}$$

$$\wedge\text{L}\ \frac{A \wedge B, A, B, \Pi \Rightarrow \Delta}{A \wedge B, \Pi \Rightarrow \Delta} \qquad \frac{\Pi \Rightarrow \Delta, A \wedge B, A \qquad \Pi \Rightarrow \Delta, A \wedge B, B}{\Pi \Rightarrow \Delta, A \wedge B}\ \wedge\text{R}$$

$$\to\text{L}\ \frac{A \to B, \Pi \Rightarrow \Delta, A \qquad A \to B, B, \Pi \Rightarrow \Delta}{A \to B, \Pi \Rightarrow \Delta} \qquad \frac{A, \Pi \Rightarrow \Delta, A \to B, B}{\Pi \Rightarrow \Delta, A \to B}\ \to\text{R}$$

**Fig. 1.** Axioms and rules for **GKcp′**.

This system differs from ordinary **GKcp** in that we do not require the $\varphi$ in the first axiom to be atomic.

**Proposition 1.** $A, \Pi \Rightarrow A \to B, \Delta$ *is provable in* **GKcp′** *iff* $\Pi \Rightarrow A \to B, \Delta$ *is provable in* **GKcp′**.

**Proposition 2.** $A, \Pi \Rightarrow A \to B, B, \Delta$ *is provable in* **GKcp′** *iff* $A, \Pi \Rightarrow A \to B, \Delta$ *is provable.*

**Theorem 1.** *Every winning strategy $\tau$ for $\Pi \vdash C, \Delta$ (i.e., for dialogue with initially disputed formula $C$ where player $O$ initially grants the formulas in $\Pi$ and disputes the formulas in $\Delta$) can be transformed into a* $\mathbf{GKcp'}$*-deduction of $\Pi \Rightarrow C, \Delta$.*

The proof of Thm. 1 is a straightforward adaptation of Fermüller's Thm. 1 [3]. Our proof that if $\varphi$ is classically valid, then there exists a winning strategy for $\varphi$ in our classical dialogue game is constructive: we will map strongly analytic $\mathbf{GKcp'}$-deductions into winning strategies.

**Definition 5.** *A* $\mathbf{GKcp'}$*-deduction is called* strongly analytic *if it contains no application of weakening, contraction, or cut.*

It is a well-known fact that:

**Lemma 1.** $\mathbf{GKcp'} \vdash \Pi \Rightarrow \Delta$ *iff there exists a strongly analytic deduction of $\Pi \Rightarrow \Delta$ in* $\mathbf{GKcp'}$ *[15].*

**Lemma 2.** *$\Pi \Rightarrow \emptyset$ is provable in* $\mathbf{GKcp'}$ *iff $\Pi \Rightarrow \perp$ is provable in* $\mathbf{GKcp'}$*.*

This brings us to the main result of this paper.

**Theorem 2.** *For every strongly analytic* $\mathbf{GKcp'}$*-deduction of $\Pi \Rightarrow \Delta$ and for every formula $\varphi$ in $\Delta$, there exists a winning strategy for the dialogue whose initial dialogue sequent is $\Pi \vdash \Delta$ and for which* **O***'s initial attack is against $\varphi$.*

*Proof.* In light of Lemma 2, we may assume that $\Delta$ is non-empty. The proof is by structural induction.

(1) The end-sequent $\Pi \Rightarrow \Delta$ of $\delta$ is an axiom because there exists a formula $A$ such that $A \in \Pi$ and $A \in \Delta$. Regardless of which formula in $\Delta$ that **O** attacks initially, it is clear that after this initial move we reach a winning state for **P**.

(2) The end-sequent $\Pi \Rightarrow \Delta$ of $\delta$ is an axiom because $\perp \in \Pi$. The existence of a winning strategy here, regardless of the formula of $\Delta$ initially attacked by **O**, is as in the previous case.

(3) The final rule application of $\delta$ is $\rightarrow$ R. That is, $\delta$ ends as follows:

$$\rightarrow \mathrm{R}\,\frac{A, \Pi \Rightarrow A \rightarrow B, B, \Delta}{\Pi \Rightarrow A \rightarrow B, \Delta}$$

We build a winning strategy for $\Pi \Rightarrow A \rightarrow B, \Delta$ and for any initially disputed formula $\varphi$ in $\{A \rightarrow B\} \cup \Delta$ as follows. The game begins with an attack by **O** on $\varphi$. The dialogue state is now $\Pi' \vdash A \rightarrow B, \Delta$, where $\Pi'$ is $\Pi$ in case $\varphi$ is not an implication and $\Pi \cup \{C\}$ in case $\varphi$ is an implication $C \rightarrow D$. **P** responds by asserting $A \rightarrow B$ (this is one of the initially disputed formulas). Since **P**'s move is a defense, by Rule E, in the next round **O** must attack this assertion by asserting the antecedent $A$. Let **P** defend against this attack by asserting $B$; we are now at an **O**-node and the dialogue state is $A, \Pi \vdash A \rightarrow B, \Delta'$. By the induction hypothesis, for this sequent we have a winning strategy $\tau$. Simply glue $\tau$ to the end of the linear order of length 4 that we have defined so far. The result is a winning strategy, because we have accounted for all possible moves by **O**.

The remaining cases (for L$\rightarrow$, $\vee$ and $\wedge$) are analogous.

## 4 Conclusion

We have provided a class of dialogue games and proved that existence of winning strategies for the Proponent in these games corresponds to classical derivability in the sequent calculus system **GKcp′**. Our proof improves on two previous results by using standard dialogue games instead of parallel dialogue games, as well as a standard classical sequent calculus.

## References


1. Clerbout, N., Gorisse, M.H., Rahman, S.: Context-sensitivity in Jain philosophy: Siddharṣigaṇi's *Commentary on the Handbook of Logic*. Journal of Philosophical Logic pp. 1–30 (2010)
2. Felscher, W.: Dialogues, strategies, and intuitionistic provability. Annals of Pure and Applied Logic 28, 217–254 (1985)
3. Fermüller, C.G.: Parallel dialogue games and hypersequents for intermediate logics. In: Mayer, M.C., Pirri, F. (eds.) TABLEAUX 2003. pp. 48–64 (2003)
4. Fermüller, C.G., Ciabattoni, A.: From intuitionistic logic to Gödel-Dummett logic via parallel dialogue games. In: Proceedings of the 33rd IEEE International Symposium on Multiple-valued Logic. pp. 188–195. IEEE Computer Society (2003)
5. Galmiche, D., Larchey-Wendling, D., Vidal-Rosset, J.: Some remarks on the relations between proofs and games. In: Bour, P., Rebuschi, M., Rollet, L. (eds.) Construction, pp. 339–357. College Publications (2010)
6. Kamlah, W., Lorenzen, P.: Logische Propädeutik: Vorschule des vernünftigen Redens. Bibliographisches Institut, Wissenschaftsverlag (1973)
7. Krabbe, E.C.W.: Formal systems of dialogue rules. Synthese 63, 295–328 (1985)
8. Lorenzen, P.: Einführung in die operative Logik und Mathematik. Springer (1955)
9. Lorenzen, P.: Logik und Agon. In: Atti del XII Congresso Internazionale di Filosofia. pp. 187–194. Sansoni (1960)
10. Rahman, S., Clerbout, N., Keiff, L.: On dialogues and natural deduction. In: Primiero, G. (ed.) Acts of Knowledge, History, Philosophy, Logic. College Publications (2009)
11. Rebuschi, M.: Implicit vs. explicit knowledge in dialogical logic. In: Majer, O., Pietarinen, A.V., Tulenheimo, T. (eds.) Games: Unifying Logic, Language, and Philosophy, pp. 229–246. Springer (2009)
12. Rückert, H.: Dialogues as a Dynamic Framework for Logic. Ph.D. thesis, Universiteit Leiden (2007)
13. Sørensen, M.H., Urzyczyn, P.: Sequent calculus, dialogues, and cut elimination. In: Barendsen, E., Geuvers, H., Capretta, V., Niqui, M. (eds.) Reflections on Type Theory, $\lambda$-Calculus, and the Mind, pp. 253–261. Universiteit Nijmegen (2007)
14. Stegmüller, W., Varga von Kibéd, M.: Strukturtypen der Logik. Springer (1984)
15. Troelstra, A.S., Schwichtenberg, H.: Basic Proof Theory. Cambridge University Press, 2nd edn. (2000)